Romanuke V.V.
Khmelnytskyy National University,
Khmelnytskyy, Ukraine

ASSERTION ON OFF-BOUND COMPONENTS IN NONREGULAR PROJECTOR OPTIMAL STRATEGY FOR A CONSTRUCTION WITH FOUR SUPPORTS WITHIN THREE IDENTICAL PARTIAL UNCERTAINTIES OF COMPRESSIONS

Actuality and essentiality of the problem

There is a known problem of distributing optimally building resources in the support construction, where each support is under partially uncertain pressure [1]. It is ever hard to before-evaluate those partial uncertainties (PU), and so they often are laid to be identical, what, on the one hand, simplifies the model of distribution, but, on another, necessitates asserting some claims to resolve the problem faster.

Analysis of recent investigations on removing partial uncertainties in constructing supports

Construction with four supports is a classic building support, where their axes are in nodes of rectangular. If the total compression on the construction is unit-normed (UN), than even within PU of compressions on each support there is a relevant model, guaranteeing the full reliability in final result. This model is an antagonistic game (AG), which kernel [1] for the classic support construction

$$T(\mathbf{X}, \mathbf{Y}) = T(x_1, x_2, x_3; y_1, y_2, y_3) = \max\{T_1(x_1, y_1), T_2(x_2, y_2), T_3(x_3, y_3), T_4(x_4, y_4)\} =$$
$$= \max\left\{\alpha\frac{x_1}{y_1^2}, \alpha\frac{x_2}{y_2^2}, \alpha\frac{x_3}{y_3^2}, \alpha\frac{1-x_1-x_2-x_3}{(1-y_1-y_2-y_3)^2}\right\} = \alpha\max\left\{\frac{x_1}{y_1^2}, \frac{x_2}{y_2^2}, \frac{x_3}{y_3^2}, \frac{1-x_1-x_2-x_3}{(1-y_1-y_2-y_3)^2}\right\} \quad (1)$$

is defined on the Cartesian product

$$\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}} = \prod_{p=1}^{2}[a_1; b_1]\times[a_2; b_2]\times[a_3; b_3] = \prod_{p=1}^{2}\left(\prod_{d=1}^{3}[a_d; b_d]\right) \subset \prod_{k=1}^{6}(0; 1) \subset \prod_{k=1}^{6}[0; 1] \subset \mathbb{R}^6 \quad (2)$$

of two parallelepipeds

$$\boldsymbol{\mathcal{X}} = [a_1; b_1]\times[a_2; b_2]\times[a_3; b_3] = \prod_{d=1}^{3}[a_d; b_d] \subset \prod_{d=1}^{3}(0; 1) \subset \prod_{d=1}^{3}[0; 1] \subset \mathbb{R}^3 \quad (3)$$

and

$$\boldsymbol{\mathcal{Y}} = [a_1; b_1]\times[a_2; b_2]\times[a_3; b_3] = \prod_{d=1}^{3}[a_d; b_d] \subset \prod_{d=1}^{3}(0; 1) \subset \prod_{d=1}^{3}[0; 1] \subset \mathbb{R}^3 \quad (4)$$

as the sets of pure strategies

$$\mathbf{X} = [x_1 \;\; x_2 \;\; x_3] \in [a_1; b_1]\times[a_2; b_2]\times[a_3; b_3] = \boldsymbol{\mathcal{X}} \quad (5)$$

and

$$\mathbf{Y} = [y_1 \;\; y_2 \;\; y_3] \in [a_1; b_1]\times[a_2; b_2]\times[a_3; b_3] = \boldsymbol{\mathcal{Y}} \quad (6)$$

of the first and second players, respectively. The value x_i is the UN pressure on the i-th support, being selected by the first player, and the value y_i is the UN cross-section square (CSS) of the i-th support, being selected by the second player, might be called the projector for further, $i = \overline{1, 3}$. The condition

$$\mu_{\mathbb{R}}([a_s; b_s]) > 0 \quad \forall\, s = \overline{1, 4} \quad (7)$$

means that every support is compressed with nonzero force, whence there is a condition

$$\sum_{i=1}^{3} b_i < 1. \quad (8)$$

Other conditions, $a_s < b_s \;\; \forall\, s = \overline{1, 4}$ and $\sum_{i=1}^{3} a_i < 1$, are consequent from (7) and (8) correspondingly. And, whatever, $\sum_{s=1}^{4} x_s = \sum_{s=1}^{4} y_s = 1$.

Purpose of the paper

Assume, that the before-evaluated partial $\{[a_i;\, b_i]\}_{i=1}^{3}$-uncertainties had appeared to be identical: $b_i = b$ and $a_i = a$ $\forall\, i = \overline{1,\,3}$. The matter is what the components of projector optimal strategy (POS)

$$\mathbf{Y}_* = \begin{bmatrix} y_1^* & y_2^* & y_3^* \end{bmatrix} \subset [a;\, b] \times [a;\, b] \times [a;\, b] = \boldsymbol{Y} \qquad (9)$$

will be, if POS appears nonregular [1]. The purpose of the current paper is to make some assertions on this.

Recollecting the convexity of AG with kernel (1) on hyperparallelepiped (2)

First above all, recall that AG with kernel (1) on hyperparallelepiped (2) is convex [1]:

$$\frac{\partial^2}{\partial y_i^2} T(x_1,\, x_2,\, x_3;\, y_1,\, y_2,\, y_3) \geqslant 0 \quad \forall\, i = \overline{1,\,3} \qquad (10)$$

as almost everywhere [1] (but minding the zero derivatives)

$$\frac{\partial^2}{\partial y_i^2}\left(\frac{x_i}{y_i^2}\right) = \frac{\partial}{\partial y_i}\left(-\frac{2x_i}{y_i^3}\right) = \frac{6x_i}{y_i^4} > 0 \quad \forall\, i = \overline{1,\,3}, \qquad (11)$$

$$\frac{\partial^2}{\partial y_i^2}\left(\frac{1 - x_1 - x_2 - x_3}{(1 - y_1 - y_2 - y_3)^2}\right) = \frac{\partial}{\partial y_i}\left(\frac{2(1 - x_1 - x_2 - x_3)}{(1 - y_1 - y_2 - y_3)^3}\right) = \frac{6(1 - x_1 - x_2 - x_3)}{(1 - y_1 - y_2 - y_3)^4} > 0 \quad \forall\, i = \overline{1,\,3}. \qquad (12)$$

Surely, (10) is also true everywhere [1]. And this convexity gives the single POS (9) with components

$$y_i^* = \frac{\sqrt{b_i}}{\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \sqrt{1 - a_1 - a_2 - a_3}} = \frac{\sqrt{b_i}}{\sum\limits_{k=1}^{3} \sqrt{b_k} + \sqrt{1 - \sum\limits_{k=1}^{3} a_k}},\; i = \overline{1,\,3}. \qquad (13)$$

However, (13) is true only if

$$\frac{\sqrt{b_i}}{\sum\limits_{k=1}^{3} \sqrt{b_k} + \sqrt{1 - \sum\limits_{k=1}^{3} a_k}} \in [a_i;\, b_i] \quad \forall\, i = \overline{1,\,3}. \qquad (14)$$

Especially this agitates the interest when compressions had been before-evaluated as three identical $[a;\, b]$-uncertainties, and it matters badly if then the component

$$y_i^* = \frac{\sqrt{b_i}}{\sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \sqrt{1 - a_1 - a_2 - a_3}} = \frac{\sqrt{b}}{3\sqrt{b} + \sqrt{1 - 3a}} \qquad (15)$$

is out of the range, that is POS appears to be off-bound (with all its components simultaneously).

Theorems on left and right off-bound (nonregular) POS

Theorem 1. In AG with kernel (1) on (2) by three identical $[a;\, b]$-uncertainties and $a \geqslant \frac{\sqrt{21} - 3}{6}$ there is POS

$$\mathbf{Y}_* = \begin{bmatrix} a & a & a \end{bmatrix}. \qquad (16)$$

Proof. Having

$$\frac{\sqrt{b}}{3\sqrt{b} + \sqrt{1 - 3a}} < a \qquad (17)$$

causes impossibility of the statement [1]

$$v_* = \alpha \frac{b_1}{(y_1^*)^2} = \alpha \frac{b_2}{(y_2^*)^2} = \alpha \frac{b_3}{(y_3^*)^2} = \alpha \frac{b}{(y_i^*)^2} = \alpha \frac{1 - a_1 - a_2 - a_3}{(1 - y_1^* - y_2^* - y_3^*)^2} = \alpha \frac{1 - 3a}{(1 - y_1^* - y_2^* - y_3^*)^2} \qquad (18)$$

with components (15) for the optimal game value v_*. However, by $y_i^* = a$ $\forall\, i = \overline{1,\,3}$ gives

$v_* = \alpha(1-3a)^{-1}$ and (16) as $\alpha \frac{b}{(y_i^*)^2} = \alpha \frac{b}{a^2} < \alpha \frac{1-3a}{(1-y_1^*-y_2^*-y_3^*)^2} = \alpha \frac{1}{1-3a}$ under maximum in (1).

Mind that here $a < b < \frac{1}{3}$ due to $3b = 1 - b_4$ and $b_4 > 0$. So,

$$\sqrt{b} < 3a\sqrt{b} + a\sqrt{1-3a},\ \sqrt{b}(1-3a) < a\sqrt{1-3a},\ \sqrt{b(1-3a)} < a,\ b(1-3a) < a^2,\ b < \frac{a^2}{1-3a},$$

$$\frac{a^2}{1-3a} \geqslant \frac{1}{3},\ \frac{3a^2}{1-3a} \geqslant 1,\ 3a^2 \geqslant 1-3a,\ 3a^2+3a-1 \geqslant 0. \tag{19}$$

The roots of the corresponding equation $3a^2 + 3a - 1 = 0$ are $a^{\langle 1 \rangle} = \frac{-3-\sqrt{21}}{6}$ and $a^{\langle 2 \rangle} = \frac{-3+\sqrt{21}}{6}$. As $a^{\langle 1 \rangle} < 0 < a^{\langle 2 \rangle} < \frac{1}{3}$, then (17) is true by $a \geqslant a^{\langle 2 \rangle}$, causing the left off-bound (nonregular) POS (16). The theorem has been proved.

Theorem 2. In AG with kernel (1) on (2) by three identical $[a; b]$-uncertainties and $b \leqslant \frac{7-\sqrt{13}}{18}$ there is POS

$$\mathbf{Y}_* = [b \quad b \quad b]. \tag{20}$$

Proof. Having

$$\frac{\sqrt{b}}{3\sqrt{b}+\sqrt{1-3a}} > b \tag{21}$$

causes impossibility of the statement (18) with components (15) for the optimal game value v_*. However, by $y_i^* = b \quad \forall\, i = \overline{1, 3}$ gives $v_* = \alpha b^{-1}$ and (20) as $\alpha \frac{b}{(y_i^*)^2} = \alpha \frac{1}{b} > \alpha \frac{1-3a}{(1-y_1^*-y_2^*-y_3^*)^2} = \alpha \frac{1-3a}{(1-3b)^2}$ under maximum in (1). So,

$$\frac{1}{3\sqrt{b}+\sqrt{1-3a}} > \sqrt{b},\ 1 > 3b + \sqrt{b(1-3a)},\ (1-3b)^2 > b(1-3a),$$

$$1-7b+9b^2 > -3ab,\ 3ab > -9b^2+7b-1,\ -9b^2+7b-1 \leqslant 0,\ 9b^2-7b+1 \geqslant 0. \tag{22}$$

The roots of the corresponding equation $9b^2 - 7b + 1 = 0$ are $b^{\langle 1 \rangle} = \frac{7-\sqrt{13}}{18}$ and $b^{\langle 2 \rangle} = \frac{7+\sqrt{13}}{18}$. As $0 < b^{\langle 1 \rangle} < \frac{1}{3} < b^{\langle 2 \rangle} < 1$, then (21) is true by $b \leqslant b_1$, causing the right off-bound (nonregular) POS (20). The theorem has been proved.

Conclusion

Conditions $a \geqslant \frac{\sqrt{21}-3}{6}$ and $b \leqslant \frac{7-\sqrt{13}}{18}$ for proved POS (16) and (20) may be restated as $a \in \left[\frac{\sqrt{21}-3}{6}; \frac{1}{3}\right)$ or $a \in \left[\frac{\sqrt{21}-3}{6}; b\right)$ and $b \in \left(0; \frac{7-\sqrt{13}}{18}\right]$ or $b \in \left(a; \frac{7-\sqrt{13}}{18}\right]$. Those ones ought to be applied for fast determining the off-bound optimal CSS within three identical PU of compressions.

References

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